§ Additional techniques of integration

$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$ $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$ $\int udv=uv-\int vdu$

1. Evaluate $\int\frac{1-2x}{1+x^2}dx$ $\arctan x-\ln(1+x^2)+C$

2. Evaluate $\int\frac{1}{1-e^x}dx$ $x-\ln(1-e^x)+C$

3. Evaluate $\int\frac{x^3-3x}{x^2-1}dx$ $\frac{1}{2}x^2-\ln(x^2-1)+C$

4. Evaluate $\int\frac{1}{\sqrt{4x-x^2}}dx$ $\sin^{-1}(\frac{x-2}{2})+C$

5. $\int_2^3\frac{1}{x^2-4x+5}dx=$ $\frac{\pi}{4}$

6. $\int x\sin xdx=$ -x cos x+ sin x +c

7. $\int \arctan xdx=$ $x\arctan x-\frac{1}{2}\ln(1+x^2)+C$

8. $\int_0^2 xe^x dx=$ e^2+1

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	-2	3	4	-1
3	2	-1	-3	5

9. The table above gives values of f, f', g, and g' for selected values of x.
If $\int_1^3 f(x)g'(x)\,dx = 8$, then $\int_1^3 f'(x)g(x)\,dx =$

-4

10. Evaluate $\int_0^{\infty} xe^{-x^2}\,dx =$ $\frac{1}{2}$

11. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx =$ π

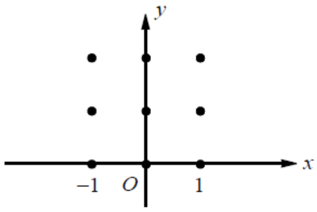
12. Find the general solution of $(x+3)y' = 2y$ $y = C(x+3)^2$

13. If y=m x+ b is a solution to the differential equation $\frac{dy}{dx} = \frac{1}{4}x - y + 1$。

m+ b=？ 1

Consider the differential equation $\frac{dy}{dx} = \frac{y^2(1-2x)}{3}$.

(a) On the axis provided sketch a slope field for the given differential equation at the nine points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

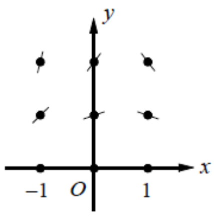
(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(\frac{1}{2}) = 4$.

Does f have a relative minimum, a relative maximum, or neither at $x = \frac{1}{2}$? Justify your answer.

(d) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $y(\frac{1}{2}) = 4$.

14.

(a)



(d) $y = \frac{3}{x^2 - x + 1}$

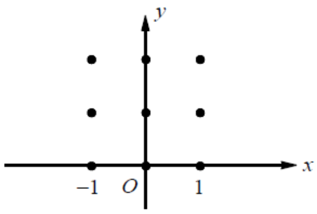
(b) $\frac{d^2y}{dx^2} = \frac{1}{3}\left[-2y^2 + (2y - 4xy)\frac{y^2(1-2x)}{3}\right]$

(c) $\left.\frac{dy}{dx}\right|_{(\frac{1}{2},4)} = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(\frac{1}{2},4)} = -\frac{32}{3} < 0$

Therefore, f has a relative maximum at $x = 1/2$.

Consider the differential equation $\frac{dy}{dx} = -2x + y + 1$.

(a) On the axis provided sketch a slope field for the given differential equation at the nine points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all the solution curves to the differential equation are concave down.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = -1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

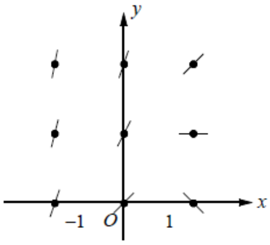
(d) Find the value of the constants m and b, for which $y = mx + b$ is a solution to the differential equation.

15.

(b) $\frac{d^2y}{dx^2} = -2x + y - 3$才對

(d)m=-2x+(m x+ b)+1 is an identity ，so m=2，b=1

(a)



(d) $m = 2$, $b = 1$

(b) $\frac{d^2y}{dx^2} = -2x + y - 1$ If the curve is CD, $y'' < 0$.

$-2x + y - 1 < 0 \Rightarrow y < 2x + 1$

Therefore, solution curves will be concave down on the half-plane below the line $y = 2x + 1$.

(c) $\left.\frac{dy}{dx}\right|_{(0,-1)} = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,-1)} < 0$. Therefore, f has a relative maximum at $(0, -1)$.

16. The number of bacteria in a culture increases at a rate proportional to the number present. If the number of bacteria was 600 after 3 hours and 19,200 after 8 hours, when will the population reach 120,000?

$t \approx 10.646$

Note that $\dot{y} = ky \Rightarrow y = Ae^{kt}$

17. A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{2}\left(3 - \frac{P}{20}\right)$, where the initial population $P(0) = 100$ and t is the time in years.

(a) What is $\lim_{t \to \infty} P(t)$?

(b) For what values of P is the population growing the fastest?

(c) Find the slope of the graph of P at the point of inflection.

(a) Write the differential equation in the standard form.

$$\frac{dP}{dt} = \frac{P}{2}\left(3 - \frac{P}{20}\right) = \frac{3P}{2}\left(1 - \frac{P}{60}\right)$$

$$\lim_{t \to \infty} P(t) = A = 60$$

(b) The population is growing the fastest when $P = \frac{A}{2}$.

$$P = \frac{A}{2} = \frac{60}{2} = 30$$

(c) The graph of P has a point of inflection at $P = \frac{A}{2}$.

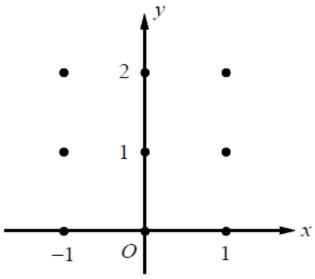
So, when $P = 30$,

$$\left.\frac{dP}{dt}\right|_{P=30} = \frac{30}{2}\left(3 - \frac{30}{20}\right) = 22.5$$

§ Euler's method and logistic models with differential equations

Consider the differential equation $\frac{dy}{dx} = 2x + y$.

(a) On the axis provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(1,1)$.



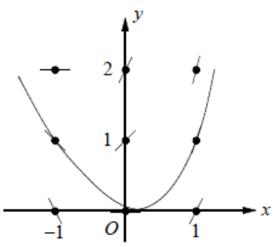
(b) Let f be the function that satisfies the given differential equation with the initial condition $f(1) = 1$. Use Euler's method, starting at $x = 1$ with a step size of 0.1, to approximate $f(1.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = -2x + b$ is a solution to the given differential equation. Show the work that leads to your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(1) = -2$. Does the graph of g have a local extremum at the point $(1, -2)$? If so, is the point a local maximum or a local minimum? Justify your answer.

1.

(a)



(b) 1.65
(c) -2
(d) g has a local minimum at $(1, -2)$.

§ Arc length and distance traveled along a smooth curve

A particle moves in the xy- plane so that its position at any time t, $0 \le t \le 4$, is given by the equations $x(t) = \cos t + t\sin t$ and $y(t) = \sin t - t\cos t$.

(a) Sketch the curve in the xy- plane for $0 \le t \le 4$. Indicate the direction in which the curve is traced as t increases.

(b) At what time t, $0 < t < 4$, does the line tangent to the path of the particle have a slope of -1?

(c) At what time t, $0 < t < 4$, does $x(t)$ attain its maximum value? What is the position $(x(t), y(t))$ of the particle at this time?

(d) At what time t, $0 < t < 4$, is the particle on the y-axis?

1.

(a)

(b) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t - (-t\sin t + \cos t)}{-\sin t + (t\cos t + \sin t)} = \dfrac{t\sin t}{t\cos t} = \tan t$

So $\dfrac{dy}{dx} = \tan t = -1 \Rightarrow t = \tan^{-1}(-1) = 3\pi/4$.

(c) $x'(t) = -\sin t + (\sin t + t\cos t) = t\cos t$

$x'(t) = 0 \Rightarrow t = \pi/2$, for $0 < t < 4$.

$x(t)$ attains its maximum value when $t = \pi/2$.

$x(\frac{\pi}{2}) = \cos\frac{\pi}{2} + \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$

$y(\frac{\pi}{2}) = \sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2} = 1$

The position when $t = \pi/2$ is $(\frac{\pi}{2}, 1)$.

(d) The particle is on the y- axis when $x(t) = 0$.

$x(t) = \cos t + t\sin t = 0$

Use a graphing calculator (in function mode) to find the value of t which makes $\cos t + t\sin t = 0$.

For $0 < t < 4$, $x(t) = 0$ when $t = 2.798$.

1 A particle moves in the xy-plane so that its position at any time t, for $0 \le t$, is given by $x(t) = e^t$ and $y(t) = 2\cos(t)$.

(a) Find the distance traveled by the particle from $t = 0$ to $t = 2$.

(b) Find the magnitude of the displacement of the particle between time $t = 0$ and $t = 2$.

1.

(a)7.035 (b)6.988

(a) $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \approx$ (b) $\sqrt{(e^2-1)^2 + (2cos2-2)^2} \approx 6.988$

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where $\frac{dx}{dt} = 1 + \cos(e^t)$ and $\frac{dy}{dt} = e^{(2-t^2)}$ for $t \ge 0$.

(a) At what time t is the speed of the object 3 units per second?

(b) Find the acceleration vector at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $1 \le t \le 4$.

2. (d) Find the magnitude of the displacement of the object over the time interval $1 \le t \le 4$.

(a) $t = 0.950$ (b) $a(2) = \left(-e^2\sin(e^2),\ \frac{-4}{e^2}\right)$ (c) 3.544 (d) 2.954

(c) $\int_1^4 \sqrt{(1+\cos e^t)^2 + (e^{2-t^2})^2}\,dt \approx 3.544$

§ Parameter equations、polar coordinates、and vector-valued functions

1. A curve is defined by the polar equation $r = 4\sin(2\theta)$ for $0 \le \theta \le \frac{\pi}{2}$.

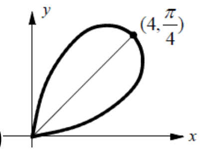
(a) Graph the curve.

(b) Find the slope of the curve at the point where $\theta = \pi/4$.

(c) Find an equation in terms of x and y for the line tangent to the curve at the point where $\theta = \frac{\pi}{4}$.

(d) Find an interval where the curve is getting closer to the origin.

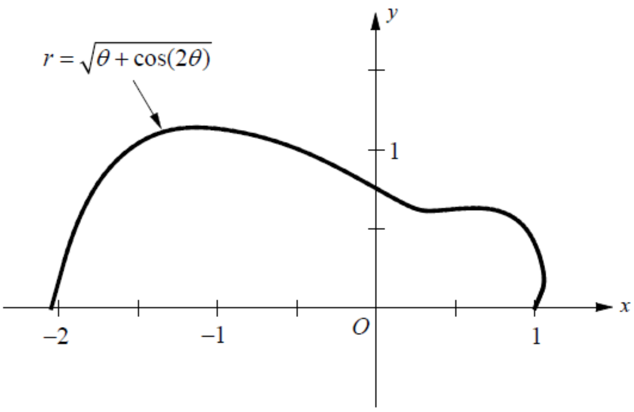
(e) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ such that the point on the curve has the greatest distance from the origin.

(a) 

(b)-1 (c) $y = -x + 4\sqrt{2}$ (d) $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ (e) $\theta = \frac{\pi}{4}$

(b) $\left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{4}} = \ldots = -1$

(d) $\frac{dr}{d\theta} < 0$



The polar curve $r=\sqrt{\theta+\cos(2\theta)}$, for $0\le\theta\le\pi$, is drawn in the figure above.

(a) Find $\frac{dr}{d\theta}$, the derivative of r with respect to θ.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate 0.5.

(c) For $\frac{\pi}{12}<\theta<\frac{5\pi}{12}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0\le\theta\le\frac{\pi}{2}$ that correspond to the point on the curve in the first quadrant with the least distance from the origin. Justify your answer.

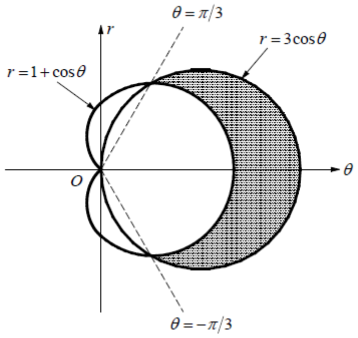
2.

(a) $\frac{dr}{d\theta}=\frac{1-2\sin(2\theta)}{2\sqrt{\theta+\cos(2\theta)}}$ (b) $\theta=0.910$

(c) r is decreasing on this interval. The curve is getting closer to the origin.

(d) $\theta=\frac{5\pi}{12}$

3. Find the area of the region that lies inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.



π

Note that $\cos 2\theta = 2\cos^2\theta - 1$

So $\int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2} d\theta$

4. The area of the shaded region that lies inside the polar curves $r = \sin\theta$ and $r = \cos\theta$ is____ $\frac{1}{8}(\pi - 2)$

$$2 \times \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} r^2 d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta d\theta =$$

5. The area of the region bounded by the polar curve $r = \theta$ and the x-axis is __ $\frac{\pi^3}{6}$

§ infinite sequences and series

The Integral Test

If f is positive, continuous, and decreasing on $[1,\infty)$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In other words:

1. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\int_1^{\infty} f(x)\,dx$ is convergent.
2. If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\int_1^{\infty} f(x)\,dx$ is divergent.

1. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

(a) convergent (b)divergent

Note that $\int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C$

***p*- Series and Harmonic Series**

The p- series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

is convergent if $p > 1$ and divergent if $0 < p \le 1$.

For $p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is called **harmonic series**.

Direct Comparison Test

Let $0 < a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive, then both series either converge or both diverge.

Note: When choosing a series for comparison, you can disregard all but the highest powers of n in both the numerator and denominator.

Determine whether the series is convergent or divergent.

1. (a) $\sum_{n=2}^{\infty} \frac{n}{n^2 - 3}$ (b) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{\sqrt{n^3} + 1}$

(a) divergent (b)convergent

$$\frac{\sin^2 n}{\sqrt[3]{n} + 1} < \frac{1}{\sqrt[3]{n} + 1} < \frac{1}{\sqrt[3]{n}}$$

Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \le a_n$, for all n greater than some integer N .

Alternating Series Estimation Theorem (Error Bound)

If S_n is a partial sum and $S = \sum_{n=1}^{\infty}(-1)^n a_n$ is the sum of a convergent alternating series that satisfies the condition $a_{n+1} \le a_n$, then the remainder $R_n = S - S_n$ is smaller than a_{n+1} , which is the absolute value of the first neglected term.

$$|R_n| = |S - S_n| \le a_{n+1}$$

Definition of Absolute and Conditional Convergence

1. $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.
2. $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converge but $\sum |a_n|$ diverges.

1. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{2n-1}$

(a)convergent (b)divergent

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt[n]{e}}{n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}n^{-2/3}$

(a) absolutely convergent (b)conditionally convergent

Note that $0 \le \frac{\sqrt[n]{e}}{n^2} \le \frac{e}{n^2} = e \times \frac{1}{n^2}$

Let $f(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+-\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$.

Use the alternating series error bound to show that $1-\frac{1}{2!}+\frac{1}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{500}$.

3.

$$f(1)=1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots+\frac{(-1)^n}{(2n)!}+\cdots$$

Since series is alternating, with terms convergent to 0 and decreasing in absolute value, the error is less than the first neglected term.

So, $\left|f(1)-(1-\frac{1}{2!}+\frac{1}{4!})\right|\le\frac{1}{6!}=\frac{1}{720}<\frac{1}{500}$.

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$.
2. $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$.
3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$.

Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty}\frac{3^n}{n!}$ (b) $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{5^n}$ (c) $\sum_{n=1}^{\infty}\frac{3^n}{2^n-1}$

1.

(a) converge (b)converge (c)diverge

Determine whether the series is conditionally convergent or absolute convergent.

2. (a) $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+3}$ (b) $\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n!}$

(a)conditionally converge (b)absolutely converge

3. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty}\frac{(-2)^n x^n}{\sqrt{n+3}}$.

$R=\frac{1}{2}$ $(-\frac{1}{2},\frac{1}{2}]$

4. What are all values of x for which the series $\sum_{n=0}^{\infty}\frac{n(x-2)^n}{3^n}$ converges?

(A) $-1<x<5$ (B) $-1<x\le 5$ (C) $-2\le x<4$ (D) $-2<x\le 4$

A

5. What are all values of x for which the series $\sum_{n=1}^{\infty} n!(3x-2)^n$ converges?

(A) No values of x (B) $(-\infty,\frac{2}{3}]$ (C) $x=\frac{2}{3}$ (D) $[\frac{2}{3},\infty)$

C

6.

The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n)!}=1-\frac{3x^2}{2!}+\frac{5x^4}{4!}-\frac{7x^6}{6!}+\cdots+(-1)^n\frac{(2n+1)x^{2n}}{(2n)!}+\cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x=0$. Give a reason for your answer.

(b) Show that $1-\frac{3}{2!}+\frac{5}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{100}$.

(c) Let g be the function given by $g(x)=\int_0^x f(t)\,dt$. Write the first four terms and the general term of the power series expansion of $\frac{g(x)}{x}$.

(a) $f'(0)=0$, $f''(0)=-3$, f has a local maximum at $x=0$ because $f'(0)=0$ and $f''(0)<0$.

(b) $\left|f(1)-(1-\frac{3}{2!}+\frac{5}{4!})\right|\le\frac{7}{6!}=\frac{7}{720}<\frac{1}{100}$ (c) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$

Lagrange Error Bound

If f has $n+1$ derivatives at c and $R_n(x)$ is the remainder term of the Taylor polynomial $P_n(x)$, then $f(x)=P_n(x)+R_n(x)$.

So $R_n(x)=f(x)-P_n(x)$ and the absolute value of $R_n(x)$ satisfies the following inequality.

$$\boldsymbol{|R_n(x)|=|f(x)-P_n(x)|\le\max\left|f^{(n+1)}(k)\right|\cdot\frac{|x-c|^{n+1}}{(n+1)!}},$$

where $\max\left|f^{(n+1)}(k)\right|$ is the maximum value of $f^{(n+1)}(k)$ between x and c.

The remainder $R_n(x)$ is called the **Lagrange Error Bound** (or **Lagrange form of the remainder**).

7.

Let $P(x)=3-2(x-2)+5(x-2)^2-12(x-2)^3+3(x-2)^4$ be the fourth-degree Taylor polynomial for the function f about $x=2$. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(2)$ and $f'''(2)$.

(b) Write the third-degree Taylor polynomial for f' about 2 and use it to approximate $f'(2.1)$.

(c) Write the fourth-degree Taylor polynomial for $g(x)=\int_2^x f(t)\,dt$ about 2.

(d) Can $f(1)$ be determined from the information given? Justify your answer.

(a) $f(2)=3$, $f'''(2)=-72$ (b) $P_3(x)=-2+10(x-2)-36(x-2)^2+12(x-2)^3$, $f'(2.1)=-1.348$

(c) $3(x-2)-(x-2)^2+\frac{5}{3}(x-2)^3-3(x-2)^4$ (d) No.

8.

Let f be the function given by $f(x)=\sin(2x)+\cos(2x)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x=0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{19} in the Taylor series for f about $x=0$.

(c) Use the Lagrange error bound to show that $\left|f(\frac{1}{5})-P(\frac{1}{5})\right|<\frac{1}{100}$

(d) Let h be the function given by $h(x)=\int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for h about $x=0$.

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots,\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$$

(a) $P(x)=1+2x-2x^2-\frac{4}{3}x^3$ (b) $\frac{(-2)^{19}}{19!}$ (c) $32\cdot\frac{0.0016}{24}=\frac{4}{1875}<\frac{1}{100}$ (d) $h(x)=x+x^2-\frac{2}{3}x^3$

(c) $|R_3(x)|=|f(x)-P_3(x)|\le \max\limits_{0\le k\le\frac{1}{5}}\left|f^{(4)}(k)\right|\times\frac{(\frac{1}{5}-0)^4}{4!}$, $f^{(4)}(x)=16\sin 2x+16\cos 2x$

$\left|f^{(4)}(x)\right|\le 16+16=32$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ..., \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$

X 用 2x 代入 得

$f(x) = \sin 2x + \cos 2x = \{2x - \frac{(2x)^3}{3!} + ...\} + \{1 - \frac{(2x)^2}{2!} + ...\} = 1 + 2x - 2x^2 - \frac{4}{3}x^3 + ...$

所以(a) $P_3(x) = 1 + 2x - 2x^2 - \frac{4}{3}x^3$

(b)f(x)在 x=0 的展開式中 x^{19} 的係數是$-\frac{(2x)^{19}}{19!}$的係數，即$-\frac{2^{19}}{19!}$

我們注意到 19 除以 4，餘數=3，是在 sin(2x)的展開式中，是負號。

(c)看前面 Lagrange Error Bound

$|R_3| = |f(x) - P_3(x)| \le \max_{c \le k \le x} |f^{(4)}(k)| \times \frac{(x-c)^4}{4!}$ ，Now c=0， $x = \frac{1}{5}$

所以$\left| f(\frac{1}{5}) - P_3(\frac{1}{5}) \right| \le 32 \times \frac{(\frac{1}{5} - 0)^4}{4!} = \frac{4}{1875} < \frac{1}{100}$

Note that $|\sin 2x| \le 1, |\cos 2x| \le 1$ ， $f^{(4)}(x) = 16\sin 2x + 16\cos 2x \le 16 + 16 = 32$

(d) $h(x) = \int_0^x f(t)dt = \int_0^x (1 + 2t - 2t^2 - \frac{4}{3}t^3 + ...)dt = x + x^2 - \frac{2}{3}x^3 - ...$