

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A , where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.
- (a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.
- (b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.
- (c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.
- (d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

ANS

(a) Riemann sum $\int_0^{10} A(h)dh \approx 176.3$

The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.

(c) $\int_0^{10} f(h)dh \approx 101.325$

(d) Chain rule $V(h) = \int_0^h f(x)dx$, $\frac{dV}{dt}\Big|_{h=5} = \dots = 1.694$

2. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \leq 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4 \ln(t^2 + 2t) \text{ for } 3 < t \leq 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

- (a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?
- (b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.
- (c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.
- (d) How many pounds of bananas are on the display table at time $t = 8$?

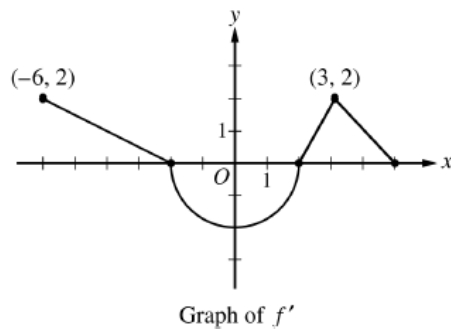
ANS

$$(a) \int_0^2 f(t) dt = 20.051$$

$$(b) f'(7) = -8.120$$

$$(c) -f(5) = -2.263 < 0$$

$$(d) 50 + \int_3^8 g(t) dt - \int_0^8 f(t) dt = 23.34$$



3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f' , the derivative of f , consists of a semicircle and three line segments, as shown in the figure above.
- Find the values of $f(-6)$ and $f(5)$.
 - On what intervals is f increasing? Justify your answer.
 - Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.
 - For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

ANS

$$(a) f(-6) = f(-2) + \int_{-2}^{-6} f'(x) dx = 3, \quad f(5) = 10 - 2\pi$$

$$(b) [-6, -2] \text{ and } [2, 5]$$

$$(c) f'(x) = 0 \quad x = -2, 2, \text{ the absolute minimum is } f(2) = 7 - 2\pi$$

$$(d) f''(-5) = -\frac{1}{2}, \quad f''(3) \text{ 不存在}$$

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius ($^{\circ}\text{C}$) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.
- (a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.
- (b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.
- (c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

ANS

(a) $y = 91 - 16t$, 43

$$\frac{d^2H}{dt^2} = -\frac{1}{4} \frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$$

$$H > 27 \text{ for } t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0 \text{ for } t > 0$$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

(c) $G(t) = 27 + \left(\frac{12-t}{3}\right)^3, 0 \leq t < 10$, 54

5. Two particles move along the x -axis. For $0 \leq t \leq 8$, the position of particle P at time t is given by $x_P(t) = \ln(t^2 - 2t + 10)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.
- (a) For $0 \leq t \leq 8$, when is particle P moving to the left?
- (b) For $0 \leq t \leq 8$, find all times t during which the two particles travel in the same direction.
- (c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.
- (d) Find the position of particle Q the first time it changes direction.

ANS

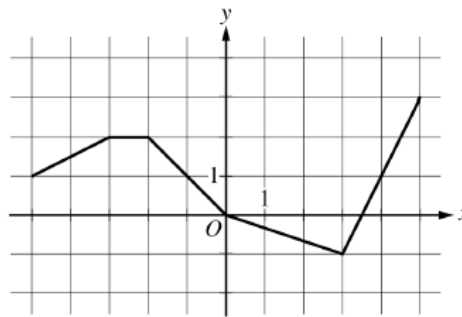
(a) $0 \leq t < 1$

(b) $1 < t < 3$ or $5 < t \leq 8$

(c) $a_Q(2) = -4$, decreasing

(d) $x_Q(3) = x_Q(0) + \int_0^3 v_Q(t) dt = 23$

x	$g(x)$	$g'(x)$
-5	10	-3
-4	5	-1
-3	2	4
-2	3	1
-1	1	-2
0	0	-3

Graph of h

6. Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x .

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

- Find the slope of the line tangent to the graph of f at $x = \pi$.
- Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.
- Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.
- Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.

ANS

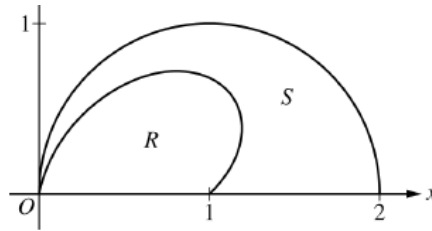
(a) $f'(\pi) = -1$

(b) chain rule $k'(\pi) = \frac{1}{3}$

(c) $m'(2) = -3$

(d)

BC



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin \theta \cos(2\theta)$ and $r = g(\theta) = 2 \cos \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x -axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x -axis.
- Find the area of R .
 - The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k .
 - For each θ , $0 \leq \theta \leq \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A , the average value of $w(\theta)$ over the interval $0 \leq \theta \leq \frac{\pi}{2}$.
 - Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

ANS

(a) 0.648

$$(b) \int_0^k ((g(\theta))^2 - (f(\theta))^2) d\theta = \frac{1}{2} \int_0^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) d\theta$$

$$(c) w_A = \frac{\int_0^{\pi/2} w(\theta) d\theta}{\frac{\pi}{2} - 0} = 0.485$$

(d) $w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.

5. Let f be the function defined by $f(x) = \frac{3}{2x^2 - 7x + 5}$.
- (a) Find the slope of the line tangent to the graph of f at $x = 3$.
- (b) Find the x -coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.
- (c) Using the identity that $\frac{3}{2x^2 - 7x + 5} = \frac{2}{2x - 5} - \frac{1}{x - 1}$, evaluate $\int_5^{\infty} f(x) dx$ or show that the integral diverges.
- (d) Determine whether the series $\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

ANS

(a) $f'(3) = -\frac{15}{4}$

(b) f has a relative maximum at $x = \frac{7}{4}$.

(c) $\int_5^{\infty} f(x) dx = \ln\left(\frac{8}{5}\right)$

f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\int_5^{\infty} \frac{3}{2x^2 - 7x + 5} dx$

(d) converges.

Or by limit comparison test

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f^{(n+1)}(0) &= -n \cdot f^{(n)}(0) \text{ for all } n \geq 1 \end{aligned}$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.
- (a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f .
- (b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.
- (c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t) dt$.
- (d) Let $P_n\left(\frac{1}{2}\right)$ represent the n th-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

ANS

(a) The general term is $\frac{(-1)^{n+1} x^n}{n}$.

(b) The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

(c) $\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \dots + \frac{(-1)^{n+1} x^{n+1}}{(n+1)n} +$

The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded by the magnitude of the first unused term, $\left| -\frac{(1/2)^5}{20} \right|$.

(d) Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20} < \frac{1}{500}$.